ANALOGS OF q-SERRE RELATIONS IN THE YANG-BAXTER ALGEBRAS

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Yang-Baxter bialgebras, as previously introduced by the authors, are shown to arise from a double crossproduct construction applied to the bialgebra

$$R_{12}T_1T_2 = T_2T_1R_{12}\,, \quad E_1T_2 = T_2E_1R_{12}\,, \quad \Delta(T) = T\hat{\otimes}T\,, \quad \Delta(E) = E\hat{\otimes}T + 1\hat{\otimes}E$$

and its skew dual, with R being a numerical matrix solution of the Yang-Baxter equation. It is further shown that a set of relations generalizing q-Serre ones in the Drinfeld-Jimbo algebras $U_q(\mathbf{g})$ can be naturally imposed on Yang-Baxter algebras from the requirement of non-degeneracy of the pairing.

1. Yang-Baxter algebras (YBA), introduced in [1, 2, 3], generalize the wide-known FRT construction [4] in the following sense: to any numerical matrix solution R of the Yang-Baxter equation there is associated a bialgebra containing the FRT one as a sub-bialgebra. Generally, this construction may provide examples of (new) bialgebras and Hopf algebras [5]. In several aspects, there is some similarity of YBA with so-called inhomogeneous quantum groups [6, 7], and also with Majid's scheme of double bosonization [8]. However, in YBA no extra (dilation) generators appear, whereas additional (analogous to q-Serre) relations are not necessarily quadratic. The main goal of the present note is to refine the concept and definition of these generalized Serre relations, first introduced in [1] and further studied in [5]. Here we obtain Serre-like relators as the elements in the kernel of some bilinear form, and the main result of the present paper is a condition (21) for such relators.

2. We recall the definition of the Yang-Baxter bialgebra. Let an invertible matrix R obey the Yang-Baxter equations $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. Consider two bialgebras Y_+ and Y_- with generators $\{u^i_j, F^i\}$, $\{t^i_j, E_i\}$, respectively, which form matrices T, U, a row E and a column F. We impose the following multiplication,

$$\begin{aligned} R_{12}\,T_1\,T_2 &= T_2\,T_1\,R_{12}\,, \quad & E_1\,T_2 &= T_2\,E_1\,R_{12}\,, \\ R_{12}\,U_1\,U_2 &= U_2\,U_1\,R_{12}\,, \quad & F_2\,U_1 &= R_{12}\,U_1\,F_2\,, \end{aligned} \tag{1}$$

comultiplication and counit, respectively:

$$\begin{array}{llll}\Delta(T)=T\hat{\otimes}T\,, & \Delta(E)=E\hat{\otimes}T+1\hat{\otimes}E\,, & \varepsilon(T)=\mathbf{1}\,, & \varepsilon(E)=0\,,\\ \Delta(U)=U\hat{\otimes}U\,, & \Delta(F)=F\hat{\otimes}1+U\hat{\otimes}F\,, & \varepsilon(U)=\mathbf{1}\,, & \varepsilon(F)=0.\end{array} \tag{2}$$

The symbol $\hat{\otimes}$ denotes the tensor product with implied matrix multiplication. There is a bilinear pairing $Y_+ \otimes Y_- \to \mathbf{C}$ satisfying

$$< U\,,T >= R\,, \quad < 1\,,T >=< U\,,1 >=< F\,,E >= \mathbf{1} \tag{3}$$

such that these are the only nonzero pairings between generators and such that Y_+ and Y_- are 'skew paired' [9]:

$$< xy\,,a >=< x\otimes y\,,\Delta(a) >\,, \quad < \Delta(x)\,,a\otimes b >=< x\,,ba >\,, \tag{4}$$

$$\varepsilon(a) =< 1,a >\,, \quad \varepsilon(x) =< x,1 > \;. \tag{5}$$

3. The bialgebras defined above form a matched pair of bialgebras, when a left action of Y_+ on Y_- and a right action of Y_- on Y_+ are suitably defined. So they can be used to build a 'double crossproduct' [9, 10] bialgebra Y. This is because there is another pairing

$$< U\,,T >^-= R^{-1}\,, \quad < 1\,,T >^-=< U\,,1 >^-= \mathbf{1}\,, \;\; < F\,,E >^-= -\mathbf{1}, \tag{6}$$

yielding a 'convolution inverse' of the pairing (3). This means,

$$< x_{(1)}\,,a_{(1)} >^-< x_{(2)}\,,a_{(2)} >=< x_{(1)}\,,a_{(1)} >< x_{(2)}\,,a_{(2)} >^-= \varepsilon(x)\varepsilon(a)\,. \tag{7}$$

In the Hopf algebra case, $< x,a >^-$ is simply $< S(x),a >$. For details, see [9]. The double crossproduct algebra Y implies additional 'cross-multiplication' relations

$$\begin{array}{ll} R_{12}\,U_1\,T_2 = T_2\,U_1\,R_{12}\,, & T_2\,F_1 = R_{12}\,F_1\,T_2\,,\\ E\,F - F\,E = T - U\,, & U_1\,E_2 = E_2\,U_1\,R_{12}\,,\end{array} \tag{8}$$

and Y_+ and Y_- are imbedded as sub-bialgebras. In [2] it is shown how an antipode can be introduced into this bialgebra, in [3] a vector space representation is constructed.

4. Commutation relations between the E_i-generators themselves, as well as between the F^i, are still missing in (1),(8). In [1, 5] two slightly different (but possibly equivalent) recipes have been proposed how to add such relations without destroying the bialgebra (respectively the Hopf algebra) structure. The first of them [1, 3] was motivated by geometrical ideas and by the desire to obtain, via appropriate choices of the matrix R, the $U_q(\mathbf{g})$ algebras. The second [5] approach was in fact an explicit solution of a bialgebra condition. However, in the present work the authors choose to propose one more approach to the problem. Now we exploit an idea which is well known in the Lie algebra theory, see e.g. [11], and has been used in [12] to recover q-Serre identities in $U_q(\mathbf{g})$: Serre relators nullify (lie in the kernel of) a suitable bilinear form.

5. Let us first recall a general fact about null ideals [10]. Let the bialgebras A and B be (skew) paired and let $I := \{a \in A \, :< B \, , a >= 0\}$. Then I is an ideal in A ('null ideal'), and can be shown to be also a bi-ideal. Indeed, let $\{e_i \, , e_\alpha\}$ and $\{e_i\}$ be bases of A and I, respectively. Then, due to (4),

$$< B \, , e_i >= 0 \, , \quad < B \otimes B \, , \Delta(e_i) >= 0 \, . \tag{9}$$

One can write down

$$\Delta(e_i) = f_i^{jk}(e_j \otimes e_k) + f_i^{\alpha k}(e_\alpha \otimes e_k) + f_i^{j\beta}(e_j \otimes e_\beta) + f_i^{\alpha\beta}(e_\alpha \otimes e_\beta) \, . \tag{10}$$

To prove that $f_i^{\alpha\beta} = 0$ we choose a set $\{e^\alpha\} \in B$ such that $< e^\alpha \, , e_\beta >= \delta^\alpha_\beta$. Then

$$0 =< e^\alpha \otimes e^\beta \, , \Delta(e_i) >= f_i^{\mu\nu} < e^\alpha \otimes e^\beta \, , e_\mu \otimes e_\nu >= f_i^{\alpha\beta} \, . \tag{11}$$

Consequently, I is a bi-ideal in the bialgebra A:

$$\Delta(I) \subset I \otimes A + A \otimes I \, . \tag{12}$$

Analogously, I is shown to be a Hopf ideal in the case of paired Hopf algebras.

6. The $\{U\}$- and $\{T\}$-subalgebras of Y_+ and Y_-, respectively, are bialgebras that are (skew) paired by (3). For general R, this pairing may be degenerate, so that 'pure-U' and 'pure-T' null bi-ideals I_+ and I_- may exist. Assume that we know both of these bi-ideals for a given R. Now let us extend these ideals to ideals I'_+ and I'_- in the full bialgebra Y by multiplying them from left and right, say

$$I \to I' := \sum Y \cdot I \cdot Y \, .$$

These extended ideals are also bi-ideals due to (12) and homomorphisity of the co-product:

$$\Delta(YIY) = \Delta(Y)\Delta(I)\Delta(Y) \subset (Y \otimes Y)(I \otimes Y + Y \otimes I)(Y \otimes Y) \subset I' \otimes Y + Y \otimes I' \, .$$

Now we divide the algebra Y by the bi-ideal $I'_+ + I'_-$ and further work with the quotient bialgebra (and the two subalgebras), keeping the notations as before. For 'regular' R [10], the double crossproduct structure is respected by the quotient.

7. However, there still might be left some degeneracy of the pairing between the (quotient) Y_+ and Y_- bialgebras. For example, elements of the form

$$E_{a_1} \ldots E_{a_N} \, \omega^{a_1 \ldots a_N} \tag{13}$$

with ω being some numerical coefficients could also nullify the bilinear form (3). Those elements (13) (and their analogs formed by F-generators) which possess this property are precisely the analogs of q-Serre relators we are interested in. Now our goal is to find all such elements and then set them to zero, thus introducing Serre-like relations on E- and F-generators.

8. Let us find out when an element of type (13) nullifies the pairing with all elements of the Y_+-bialgebra. Consider first the case $N = 2$, and begin with the following computation:

$$< F^m F^n \,, E_i E_j \omega^{ij} >=< \Delta(F^m)\Delta(F^n) \,, E_j \otimes E_i > \omega^{ij}$$

$$=< (F^m \otimes 1 + u^m_p \otimes F^p)(F^n \otimes 1 + u^n_q \otimes F^q) \,, E_j \otimes E_i > \omega^{ij}$$

$$=< F^m u^n_q \otimes F^q + u^m_p F^n \otimes F^p \,, E_j \otimes E_i > \omega^{ij} =< F^m u^n_i + u^m_i F^n \,, E_j > \omega^{ij}$$

$$= (< F^m \otimes u^n_i \,, E_k \otimes t^k_j > + < u^m_i \otimes F^n \,, 1 \otimes E_j >)\omega^{ij} = (< u^n_i \,, t^m_j > + \delta^m_i \delta^n_j)\omega^{ij}$$

$$= (R^{nm}_{ij} + \mathbf{1}^{mn}_{ij})\omega^{ij} = (B + \mathbf{1})^{mn}_{ij}\omega^{ij} = ([2!]_B \, \omega)^{mn} \,. \tag{14}$$

We used here the standard definition of the 'braid matrix' B:

$$B^{ij}_{mn} := R^{ji}_{mn} \,, \quad B_m := B_{m,m+1} \,, \quad B_1 B_2 B_1 = B_2 B_1 B_2 \,. \tag{15}$$

Definition and properties of 'braided factorial' $[N!]_B$ can be found in [1, 3, 13].

Now, consider a general element $F^m F^n \Phi(u)$ of the bialgebra Y_+, which is relevant in the $N = 2$ case (here $\Phi(u)$ may, of course, carry its own indices):

$$< F^m F^n \Phi(u) \,, E_i E_j > \omega^{ij} =< F^m F^n \otimes \Phi(u) \,, \Delta(E_i)\Delta(E_j) > \omega^{ij}$$

$$=< F^m F^n \,, E_p E_q >< \Phi(u) \,, t^p_i t^q_j > \omega^{ij} =< \Phi(u) \,, (\mathbf{1} + B)^{mn}_{pq} \, t^p_i t^q_j > \omega^{ij}$$

$$=< \Phi(u) \,, t^m_p t^n_q > (\mathbf{1} + B)^{pq}_{ij}\omega^{ij} =< \Phi(u) \,, t^m_p t^n_q > ([2!]_B \, \omega)^{pq} \,, \tag{16}$$

where the relation

$$B_{12} T_1 T_2 = T_1 T_2 B_{12} \tag{17}$$

has been used. Thus, we obtain

$$[2!]_B \, \omega = 0 \tag{18}$$

as a necessary and sufficient condition for $E_i E_j \, \omega^{ij}$ to lie in the null ideal.

In the general case of arbitrary N in (13), we find:

$$< F^{q_1} \ldots F^{q_N} \,, E_{a_1} \ldots E_{a_N} >=< \Delta(F^{q_1}) \ldots \Delta(F^{q_N}) \,, E_{a_N} \otimes E_{a_1} \cdot \ldots \cdot E_{a_{N-1}} >$$

$$= \sum_{k=1}^{N} < u^{q_1}_{p_1} \ldots u^{q_{k-1}}_{p_{k-1}} F^{q_k} u^{q_{k+1}}_{p_{k+1}} \ldots u^{q_N}_{p_N}, E_{a_N} >< \ldots F^{p_{k-1}} F^{p_{k+1}} \ldots, E_{a_1} \ldots E_{a_{N-1}} >$$

$$= \sum_{k=1}^{N} B^{q_k q_{k+1}}_{p_k b_{k+1}} B^{b_{k+1} q_{k+2}}_{p_{k+1} b_{k+2}} \ldots B^{b_{N-1} q_N}_{p_{N-1} a_N} < F^{q_1} \ldots F^{q_{k-1}} F^{p_k} \ldots F^{p_{N-1}}, E_{a_1} \ldots E_{a_{N-1}} > \,. \tag{19}$$

By induction, using recursion formulas for braided factorials [1, 13], we come to

$$< F^{q_1} \ldots F^{q_N} \,, E_{a_1} \ldots E_{a_N} >= ([N!]_B)^{q_1 \ldots q_N}_{a_1 \ldots a_N} \tag{20}$$

(compare similar formulas in [8]). So, the null ideal condition

$$[N!]_B\omega = 0 \tag{21}$$

is proved, because a proper generalization $F \dots F\Phi(u)$ is easily handled in complete analogy with (16).

Quite analogously, it can be shown that elements of the form

$$\eta_{a_1\dots a_N}\, F^{a_1} \dots F^{a_N} \tag{22}$$

lie in the kernel of (3) if and only if

$$\eta\, [N!]_B = 0\,. \tag{23}$$

9. We now consider the most general null element containing E generators. Explicitly using the relation $ET = TER$, we can cast it to the form

$$\omega^{[\alpha]}(T)E_{[\alpha]} := \omega^{\alpha_1\dots\alpha_N}(T)E_{\alpha_1}\dots E_{\alpha_N}\,, \tag{24}$$

allowing polynomial dependence of the coefficients ω on the generators T. Then a condition of being in the kernel is derived as follows:

$$< \Phi(U)F^{[\gamma]}\,, \omega^{[\alpha]}(T)E_{[\alpha]} >=< \Phi(U)\otimes F^{[\gamma]}\,, \Delta(\omega^{[\alpha]}(T))(1\otimes E_{[\alpha]}) >$$

$$=< \Phi(U)\,, \omega^{[\alpha]}_{(1)}(T) >< F^{[\gamma]}\,, \omega^{[\alpha]}_{(2)}(T)E_{[\alpha]} >$$

$$=< \Phi(U)\,, \omega^{[\alpha]}_{(1)}(T) >< \Delta(F^{[\gamma]})\,, E_{[\alpha]}\otimes \omega^{[\alpha]}_{(2)}(T) >$$

$$=< \Phi(U)\,, \omega^{[\alpha]}_{(1)}(T) > ([N!]_B)^{[\gamma]}_{[\alpha]} < 1\,, \omega^{[\alpha]}_{(2)}(T) >$$

$$=< \Phi(U)\,, ([N!]_B)^{[\gamma]}_{[\alpha]}\omega^{[\alpha]}(T) >= 0\,, \tag{25}$$

where $\omega_{(1)}\otimes\omega_{(2)} := \Delta(\omega)$, and the formula (20) in the form $< F^{[\gamma]}\,, E_{[\alpha]} >= ([N!]_B)^{[\gamma]}_{[\alpha]}$ has been used. The equality (25) must be fulfilled for arbitrary $\Phi(U)$. Consequently, due to non-degeneracy of the $T-U$-pairing, the relation

$$[N!]_B\omega(T) = 0 \tag{26}$$

is a necessary and sufficient condition for the relator (24) to lie in the kernel of the bilinear form $<\,,\,>$.

10. We will finally show that the (remaining) null bi-ideals are already generated by the elements (13) and (22) satisfying (21) and (23), respectively. In other words, considering the T-dependent case (24),(26) does not enlarge the set of relators (13) obtained with the use of all solutions of the T-independent condition (21). Consider an element $\omega(T) \in V\otimes Y_-(T)$ satisfying (26), where V is the vector space associated to the multi-indices $[\alpha]$ appearing above and $Y_-(T)$ is the subalgebra in Y_- generated by the T generators. Now the linear map $[N!]_B : V\otimes Y_-(T) \to V\otimes Y_-(T)$ is induced by $[N!]_B : V \to V$, such that the

kernel of the first map is $K \otimes Y_-(T)$, K being the kernel of the second one. So, given all solutions of eq. (21) and the corresponding q-Serre relators (13), we do not enlarge the null ideal by considering also the T-dependent case (26).

11. Let us summarize the main steps in the construction of bialgebras (and Hopf algebras) using the method proposed here. We begin with specifying any invertible matrix solution R of the Yang-Baxter equation (for introducing an antipode, an additional 'skew' invertibility [2, 4] of R would be required). Then we build bialgebras (1),(2) (which may, under suitable conditions, be extended to Hopf algebras [2]) being paired via (3),(6). The double crossproduct construction is applied to these bialgebras in order to obtain the 'cross-multiplication' relations (8). Now we find 'pure T, U' null ideals (they are specific to a given R), extend them as described in Sect. 6 and divide out by the resulting bi-ideals. The next step is to solve eq. (21), find all elements of type (13) and their F-analogs (22) in explicit form, and then equate them to zero. This yields (if any solutions of (21) exist for a given R) a set of generalized q-Serre relations which, for appropriate R, coincide with known q-Serre relations of $U_q(\mathbf{g})$. Several examples of this procedure were presented in our papers [1, 3, 5].

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